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THE MODELLING OF CYCLIC FILTRATION AND VARIABLE-RATE FILTRATION†

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The principal relations of cyclic filtration and variable-rate filtration are established within the framework of a generalized mathematical model, the prototype of which is the Shekhtman model [1]. © 2002 Elsevier Science Ltd. All rights reserved.

As a rule, existing mathematical models of filtration through porous media do not enable one to describe cyclic filtration and variable-rate filtration which are extensively used in industry. At the same time, in order to achieve this aim, it is sufficient to employ generalized mathematical models of the minimum necessary degree of generality, that is, in the case being considered here, to take account, in existing models, of the dependence of the inlet concentration on time, the initial distribution of the deposit, the existence of a transient state of limiting saturation of the charge by the deposit and the dependence of the coefficients on the filtration rate. (An important exception to what has been said above is the initial stage of magnetic cyclic filtration, that is, the stage up to the processing of the inlet cross-section of the charge when the concentration at the filter outlet does not depend on time [2]).

1. A VARIABLE FILTRATION RATE

The generalized Shekhtman mathematical model, with appropriate additional conditions in the case being considered, has the form

$$\frac{\partial\rho}{\partial t}+v(t)\frac{\partial C}{\partial x}=0, \quad \frac{\partial\rho}{\partial t}=\beta(t)C\left[1-\frac{\rho}{\rho_*(t)}\right] \tag{1.1}$$

$$C|_{x=0}=C_0, \quad \rho|_{t=0}=\rho_0(x) \tag{1.2}$$

Here, x is the coordinate in the direction of filtration ($0 \leqslant x \leqslant L$, L is the height of the charge), $v(t)$ is the filtration rate, $\beta(t)$ is the kinetic coefficient, $C(x, t)$ is the concentration of the particles suspended in the pores of the charge, C_0 = const is their concentration at the filter inlet, $\rho(x, t)$ is the concentration of the deposit in the charge and $\rho_*(t)$ is the limiting density of saturation of the charge by the deposit at the instant of time t [3]

$$\rho_*(t)=\beta(t)C_0/a(t), \qquad a(t)=a_0v(t), \qquad a_0=\text{const}$$

On eliminating the concentration ρ from system (1.1), we obtain an equation which, after integrating with respect to x taking account of the first condition of (1.2), leads to the Bernoulli equation

$$\frac{\partial C}{\partial t}+\frac{a_0}{C_0}\left\{x\frac{\partial\rho_*(t)}{\partial t}-v(t)C_0\right\}C=-\frac{a_0v(t)C^2}{C_0} \tag{1.3}$$

In order to solve this equation, we will first find the corresponding initial condition from system (1.1). This initial condition has the form

$$C|_{t=0}=\frac{C_0}{E_1(x)}, \quad E_1(x)=\exp\left\{\frac{\beta(0)}{v(0)}\int_0^x\left[1-\frac{\rho_0(x')}{\rho_*(0)}\right]dx'\right\} \tag{1.4}$$

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Solving Eq. (1.3) with initial condition (1.4), we obtain

$$C(x,t) = C_0 E_2(x,t) E_3(x,t) \tag{1.5}$$

$$E_2(x,t) = \exp\left\{a_0 \int_0^t v(\tau)\,d\tau + \frac{a_0 x}{C_0}[\rho_*(0) - \rho_*(t)]\right\}, \quad E_3(x,t) = \left[E_1(x) + a_0 \int_0^t v(\tau) E_2(x,\tau)\,d\tau\right]^{-1}$$

System (1.1) is non-linear with respect to the concentration ρ, and hence the possibility of solving it by eliminating the concentration C is problematic. However, ρ can be found from the second equation of (1.1) in which the derivative $\partial\rho/\partial t$ is now replaced by the known function $(-v(t)\ \partial C/\partial x)$ from Eq. (1.1). Hence, using expression (1.5), we shall have

$$\rho(x,t) = \rho_*(t) + \frac{C_0}{a_0}\frac{\partial \ln C}{\partial x} = [\rho_0(x) E_1(x) + a_0 \int_0^t v(\tau)\rho_*(\tau) E_2(x,\tau)\,d\tau] E_3(x,t) \tag{1.6}$$

Expressions (1.5) and (1.6) contain a whole volume of information on the process of variable-rate filtration through homogeneous charges, the mass transfer in which occurs in accordance with the laws which have been adopted in models (1.1) and (1.2). These expressions are sufficient for solving a number of optimization problems in the case of variable-rate filtration, and not only for finding the times of the protective action of the filter and the times taken to reach the limiting pressure losses to which one is usually confined in filtration theory.

In the few papers on the theory of filtration with a decreasing rate, this rate is approximated, as a rule, by functions which enable one to obtain solutions acceptable for applied calculations. For example, the required decrease in the filtration rate has been approximated by a step function [4] and the possibility of approximating the linear law for its decrease by a hyperbolic law under actual operating conditions for filters has been demonstrated in [5].

Filtration at a variable rate in the direction of the flow occurs in filters with a changing geometry. However, this is a special case of mass transfer which is more general than the case considered. Actually, it can be shown (see [6]) that constancy of the cross-section of the charge is implicitly assumed in Eqs (1.1).

2. CYCLIC FILTRATION

In accordance with the actual conditions for filters operating under cyclic conditions, it is assumed that the volume of the liquid that is filtered is significantly greater than the working volume of the filter. For the same reasons, it is assumed that the filtration rate is constant and that the charge of the filter is free from a deposit at the initial instant of time.

We will change to dimensionless variables, using $\rho_*/(\beta C_0)$, v/β, C_0, ρ_* as the scales of time, length, concentration of the suspended particles and concentration of the deposit, respectively. The previous notation is retained for the new variables.

The initial system of equations for the ith cycle ($i = 1, 2, \ldots, n$) in the dimensionless variables and with the corresponding additional conditions has the form

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial C_i}{\partial x} = 0, \quad \frac{\partial \rho_i}{\partial t} = C_i(1 - \rho_i) \tag{2.1}$$

$$C_0 = 1, \quad C_{0i}(t) = C_i(L,t), \quad \rho_0(x,0) \equiv 0, \quad \rho_i(x,0) \equiv \rho_{i-1}(x,t_0) \tag{2.2}$$

Here t_0 is the duration of a single filtration cycle, $C_i(x, t)$ is the normalized concentration of the suspended particles in the charge within the limits of the ith cycle, $C_0 = 1$ is the normalized initial concentration of these particles in the liquid being filtered or their concentration at the filter inlet during the first filtration cycle, $C_0(t)$ is the normalized concentration of the suspended admixed particles at the filter inlet within the limits of the $(i + 1)$th cycle, $C_i(L, t)$ is their normalized concentration at the filter outlet within the limits of the ith cycle, $\rho_i(x, t)$ is the normalized concentration of the deposit which has accumulated in the charge after a time $[(i - 1)t_0 + t]$, and $\rho_0(x, 0)$ is the normalized distribution of the deposit in the charge at the beginning of the first cycle.

For each but first cycle, the input concentration is the output concentration of the preceding cycle and the initial distribution of the deposit is its distribution after the time t_0 of the duration of the preceding cycle. The time taken for the liquid to pass through the filter can be neglected since, in practice, the volume of this liquid is significantly greater than the working volume of the filter.

Solving problem (2.1), (2.2) in accordance with the algorithm presented in Section 1, we obtain

$$\begin{aligned}
&C_i(x,t)=C_{i-1}(L,t)Z_{i-1}(L,t)\Phi(x,L,t,t_0)\\
&\rho_i(x,t)=[Z_{i-1}(L,t)-1+\rho_{i-1}(x,t_0)R_0(x,t_0)]\Phi(x,L,t,t_0)\\
&\Phi(x,L,t,t_0)=[R_0(x,t_0)+Z_{i-1}(L,t)-1]^{-1}\\
&R_0(x,t_0)=\exp\{\int_0^x[1-\rho_{i-1}(x',t_0)]dx'\},\qquad Z_{i-1}(L,t)=\exp[\int_0^t C_{i-1}(L,\tau)d\tau]
\end{aligned}\tag{2.3}$$

From this, we find, in accordance with conditions (2.2), the concentrations $C_1(x, t)$ and $\rho_1(x, t)$ in the first filtration cycle

$$\begin{aligned}
&C_1(x,t)=\frac{e^t}{e^t+A_1(x)},\qquad \rho_1(x,t)=\frac{B_1(t)}{e^x+B_1(t)}\\
&A_1(x)=e^x-1,\qquad B_1(t)=e^t-1
\end{aligned}\tag{2.4}$$

In order to find the concentrations $C_2(x, t)$ and $\rho_2(x, t)$ in the second filtration cycle, we substitute the expressions for the input concentration $C_1(L, t)$ and the initial distribution of the deposit $\rho_1(x, t_0)$ into relations (2.3). We obtain

$$\begin{aligned}
&C_2(x,t)=\frac{e^t}{e^t+A_2(x)},\qquad \rho_2(x,t)=\frac{B_2(t)}{e^x+B_2(t)}\\
&A_2(x)=e^{L-t_0}(e^x+e^{t_0}-1)-1,\qquad B_2(t)=e^{t_0-L}(e^t+e^L-1)-1
\end{aligned}\tag{2.5}$$

In a similar way, we find the concentrations $C_n(x, t)$ and $\rho_n(x, t)$ in the nth filtration cycle

$$C_n(x,t)=\frac{e^t}{e^t+A_n(x)},\qquad \rho_n(x,t)=\frac{B_n(t)}{e^x+B_n(t)}\tag{2.6}$$

Here,

$$\begin{aligned}
&A_1(x)=e^x-1,\quad A_{n258}(x)=\frac{1+A_{n-1}(L)}{1+B_{n-1}(t_0)}[e^x+B_{n-1}(t_0)]-1\\
&B_1(t)=e^t-1,\quad B_n(t)=\frac{1+B_{n-1}(t_0)}{1+A_{n-1}(L)}[e^t+A_{n-1}(L)]-1;\qquad n=2,3,\ldots
\end{aligned}$$

The positiveness of the functions $A_i(x)$ and $B_i(t)$ $(i = 2, 3, \ldots, n)$ for arbitrary x and t, the positiveness of the functions $A_1(x)$ and $B_1(t)$ when $0 < x \leqslant L$, $0 < t \leqslant t_0(A_i(0) = 0, B_i(0) = 0)$ and, also, the inequalities

$$A_{i+1}(x) > A_i(x),\qquad B_{n+1}(t) > A_n(t)$$

follow from expressions (2.3)–(2.6).

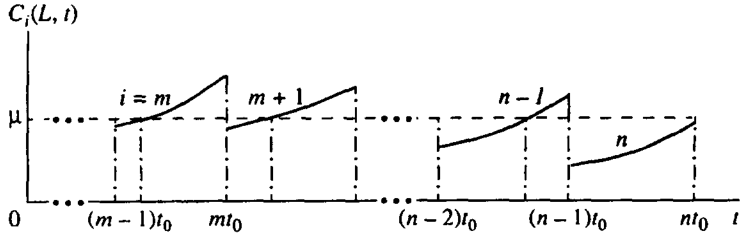

Fig. 1

It can therefore be concluded that

$$\partial C_i(x,t)/\partial t > 0, \qquad \partial \rho_i(x,t)/\partial x < 0 \text{ when } 0 < x < L \text{ and } 0 < t < t_0$$

Consequently, within the limits of a single cycle, the concentration $C_i(x, t)$ is a monotonically increasing function of time and the concentration $\rho_i(x, t)$ is a monotonically decreasing function of x when t = const. However, for fixed x and t, the concentration $C_i(x, t)$ decreases and the concentration $\rho_i(x, t)$ increases as the number of filtration cycles (i) increases.

Compared with simple filtration, cycle filtration has basic characteristics as a consequence of which the need arises to generalize certain basic characteristics of the filtration process and to introduce new characteristics. This applies, first, to the time of the protective action of the filter t_p, instead of which it is advisable to introduce a cyclic clearing (cleaning) time of the liquid being filtered, t_c, up to a previously specified level $\mu(0 < \mu < 1)$. Actually, in the case of simple filtration and a constant input concentration, the concentration at the filter outlet $C(L, t)$ increase continuously. The time t_p was therefore defined as the time taken to attain the limiting admissible level μ of the concentration of the admixed particles in the liquid being filtered, after which the filtration was stopped. In the case of cyclic filtration being considered, the concentration at the filter outlet $C(L, t)$ has the form of a sequence of curvilinear trapezia with a height which decreases as the cycle number increases (Fig. 1). In other words, the situation is fundamentally different here and, in particular, because, during the whole time up to when filtration ceases or a significant part of it, the concentration at the filter outlet $C(L, t)$ exceeds the permissible level. It is therefore natural to define the time t_c as the time from when the quality of the filtrate in subsequent filtration is no worse than what is required. This means that the time t_c is defined differently depending on the intended destination of the filtrate (like the time t_p in the case of a variable input concentration). In fact, situations are possible in the general case under cyclic conditions when the level μ is attained at different instants of time t during a quite considerable number of cycles (Fig. 1). As a rule, the requirement of the consumer regarding the filtrate quality is either a requirement concerning the average value of the output concentration $C_i(L, t)$ after a time t_0 or a different kind of constraint on the characteristics of the overshoots in this concentration above the level μ after a specified period of time. It is therefore best to shut down the filtration process long before the nth cycle, starting from which the curve $C_n(L, t)$ sinks completely below the level μ. It is thereby possible to avoid excessive cleaning of the liquid being filtered and to increase the filter output.

Apart from the time t_c, the time at which the limiting pressure losses are reached, t_n, is also one of the number of basic characteristics of the cyclic filtration process. To calculate it is necessary to establish how the pressure drop on the charge of the filter $h_n(t)$ increases under the operating conditions being considered. In the case of filters of constant cross-section with an arbitrary homogeneous charge, we have [3]

$$h_n(t) = i_0 \varepsilon_0^3 \int_0^L \frac{dx}{\varepsilon_n^3(x,t)}, \qquad \varepsilon_n(x,t) = \varepsilon_0\left(1 - \frac{\rho_n}{\gamma\varepsilon_0}\right) \tag{2.7}$$

where i_0 = const is the initial hydraulic gradient, ε_0 = const is the initial porosity of the charge, $\varepsilon_n(x, t)$ is the porosity of the charge in a cross-section x at the instant of time t within the limits of the nth filtration cycle and γ is the density of the deposit which is formed. Changing to the dimensionless variables x, t, ρ_n, $z = e^x + B_n(t)$ with the scales introduced earlier and the dimensionless variable $h_n(t)$ with the scale $i_0 v/\beta$, while retaining the previous notation for these variables, and substituting expression (2.6) for ρ_n into (2.7), we have

$$h_i(t) = \sum_{k=0}^{6} q_k(t) B_n^{6-k}(t)$$

$$r_0 = r_1 = r_2 = r_3 = (1-p)^3, \qquad r_4 = r_6 = 1, \qquad r_5 = 1 - 3p; \qquad p = \rho_*/(\gamma\varepsilon_0)$$

$$q_0 = Lr_0, \qquad q_k(t) = \frac{r_k}{k}[M^k(t) - N^k(t)]$$

$$M^k(t) = [e^L + B_n(t)]^k, \qquad N^k(t) = 1 + B_n(t), \qquad k = 1, 2, \ldots, 6$$

Hence, the time t_h is determined for a specified hydraulic pressure.

The expressions obtained above are sufficient to optimize the operation of filters of the type which has been considered working under cyclic conditions.

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